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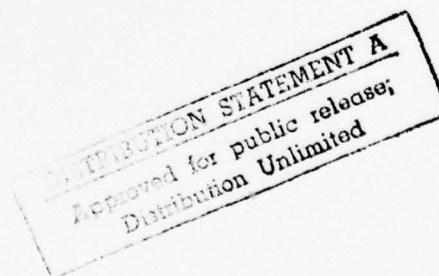
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NEW LIGHT ON WEIBULL THEORY

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In 1939 Weibull applied weakest link theory to the interpretation of the variability of fracture stress of nominally identical specimens. He attributed this variation to the presence of sources of weakness, or flaws, that were given a strength, but no other physical properties. Accordingly, his theory rests primarily on the statistics of extreme values.

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ABSTRACT

In 1939 Weibull applied weakest link theory to the interpretation of the variability of fracture stress of nominally identical specimens. He attributed this variation to the presence of sources of weakness, or flaws, that were given a strength, but no other physical properties. Accordingly, his theory rests primarily on the statistics of extreme values.

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INTRODUCTION

In 1939,[1-2] Weibull introduced weakest link theory into mechanics with the primary purpose of accounting for experimentally observed variations in the fracture stress of nominally identical specimens. He showed how test data on laboratory specimens uniformly stressed in simple tension can be used to find the failure statistics of specimens of different size and nonuniform loading. By assuming that only tension contributes to fracture, he extended the theory to apply also to bending, and in fact to any uniaxial combination of bending and tension. Because of the essentially rational nature of the theory and its generally good agreement with experiment, it has become widely known and used. For convenience this body of theory will be referred to herein as Theory A.

Weibull also gave a procedure for finding the probability of failure for finding the probability of failure for polyaxial stress states. This procedure, which we shall call Theory B, requires a treatment of uniaxial tension somewhat different from that employed in Theory A. For instance, the number of weak spots per unit volume found using Theory B differs from that found using Theory A.

Theory B has been questioned by a number of investigators. Barnett, et al.[3] considered Weibull's rule for polyaxial tension to be plausible but not necessarily correct, and listed five similar procedures they considered equally plausible. Some investigators who accept Theory A choose not to use Theory B, and employ some other approach to polyaxial stress states, such as the assumption of independence of principal stresses.[4-6] It may be that Weibull came to doubt Theory B himself since he wrote in 1966,[7] "Another problem of a more theoretical nature will be to deduce the effect of bi- and triaxiality on the distribution functions of one-dimensional stresses. If the principal stresses are acting independently of each other, and it seems that such materials may exist, then the problem may be soluble along lines previously sketched. In other cases the solution is very intricate and will certainly require close examination of the physical behavior of the material in question."

Weibull assumed that the fundamental cause of the variations in fracture stress was the presence of weak spots or flaws in the material.

He suggested that they might be due to minute   foreign bodies, impurities, or cracks. The flaws were given strengths, but no other properties. Thus Weibull theory is based mainly on extreme value statistics.

Ideally, statistical fracture theory should also give consideration to fracture mechanics and material microstructure. Progress made along these lines in recent years provides fresh insight into Weibull's theories. The primary objectives of the present paper are to outline a more general and fundamental treatment of the fracture statistics of a solid containing randomly oriented microcracks, and to show how it reduced to Weibull's results when certain simplifying assumptions are made.

FUNDAMENTAL EQUATIONS OF WEAKEST LINK THEORY

Let it be assumed that a stressed solid can fail due to any of a number of independent and mutually exclusive mechanisms or causes, each having an infinitesimal probability of failure $(\Delta P_f)_i$. The probability that the i'th mechanism will _not_ cause failure is $(P_s)_i = 1 - (\Delta P_f)_i$. The overall probability of survival is the product of the individual probabilities of survival, i.e.

$$P_s = \prod_i (P_s)_i = \prod_i \left\{ 1 - (\Delta P_f)_i \right\}$$

$$\cong \prod_i \exp\left[- (\Delta P_f)_i \right] = \exp\left[- \sum_i (\Delta P_f)_i \right] . \tag{1}$$

The sum of the individual probabilities of failure appearing in the final equality above was called by Weibull the "risk of rupture" and was given the symbol B. Note that the "risk of rupture" so defined can be arbitrarily large, but regardless of its value $0 \le P_s \le 1$, as of course it must on physical grounds.

Let us assume that the potential causes of failure are many individual cracks. For purposes of analysis it is convenient to group the cracks according to location, the applied stress state, and crack critical stress. We assume that the stress state varies slowly so that within a volume element ΔV all cracks will be subject to essentially the same macroscopic stress. We also assume that the material is macroscopically homogeneous, so that a function $N(\sigma_c)$ depending only on the material can be defined as the number of cracks per unit volume having a critical stress equal to or less than σ_c. When $N(\sigma_c) < 1$, we can regard it as the probability that a crack of critical stress $\le \sigma_c$ will be found in a unit volume. The critical stress of a crack is defined as the remote stress which will cause fracture when applied normal to the crack plane. The probability that a crack having a critical stress in the range σ_c to $\sigma_c + d\sigma_{cr}$ exists in volume element ΔV is then $\Delta V[dN(\sigma_c)/d\sigma_c]d\sigma_c$.

If such a crack is present, the probability that it will fracture depends on its orientation, the stress state, and the fracture criterion. We assume that there is a solid angle Ω such that a crack will fracture if and only if its normal lies within Ω. This means that if the normal lies within Ω, $\sigma_e > \sigma_c$, where σ_e is the effective stress corresponding to the fracture criterion selected. If the cracks are randomly oriented, the probability that a crack will fracture under the applied stress Σ is $\Omega(\Sigma, \sigma_c)/4\pi$.

The probability of failure due to a crack in the critical stress range $d\sigma_c$ located in volume element ΔV is the product of the above probabilities, i.e.,

$$\left(\Delta P_f\right)_i = \left(\Delta V \frac{dN(\sigma_c)}{d\sigma_c} d\sigma_c\right)\left(\frac{\Omega(\Sigma, \sigma_c)}{4\pi}\right) \quad . \tag{2}$$

Substituting equation (2) into equation (1) and changing sums into integrals, we obtain

$$P_s = 1 - P_f = \exp\left[-\int dV \int d\sigma_c \frac{dN}{d\sigma_c} \frac{\Omega}{4\pi}\right] \quad . \tag{3}$$

Now, if we assume that the flaws are isotropic as, for instance, spherically distributed local impurities would be, equation (3) becomes much simpler. In simple tension σ, $\Omega/4\pi$ becomes 0 or 1 depending on whether σ is less or greater than σ_c. Over the range $0 < \sigma_c < \sigma$, $\Omega/4\pi = 1$, so

$$\int_0^\sigma d\sigma_c \frac{dN}{d\sigma_c} \frac{\Omega}{4\pi} = N_i(\sigma) \quad , \tag{4}$$

where N_i is the density of isotropic flaws. Thus,

$$P_f = 1 - \exp\left[-\int dV \, n(\sigma)\right] \quad , \tag{5}$$

where we have changed $N_i(\sigma)$ to $n(\sigma)$ to agree with Weibull's notation. Equation (5) is identical to that employed by Weibull in Theory A.

Accordingly, the simplest interpretation of Theory A is to say it is based on the assumption that the flaws are isotropic.

There is another possible interpretation of equation (5). This is to permit the flaws to be cracks but take $n(\sigma)$ to be the number of cracks per unit volume that will be fractured by a uniaxial stress σ applied to the specimen. Some cracks with an intrinsic strength considerably less than σ will survive because their crack planes are parallel or nearly parallel to the tensile axis. Thus $n(\sigma)$ is smaller than our $N(\sigma)$, which is defined to be the number of cracks per unit volume having an intrinsic strength equal to or less than σ. We can also regard $N(\sigma)$ as the number of cracks per unit volume that would be fractured by an equitriaxial tension equal to σ.

Returning now to the assumption that flaws are cracks, the probability of survival can be found using equation (3) provided we know $N(\sigma_c)$ and Ω. The latter is a function of stress state, the critical stress of the crack under consideration, and the assumptions made concerning the form of the effective stress σ_e. In simple tension and equibiaxial tension, analytical expressions can be found for Ω.[8] In the general case, we find Ω by integrating $d\Omega$ over the range in which $\sigma_e > \sigma_c$. One way of accomplishing this is to integrate over the entire angular range but include an operator which goes to zero when $\sigma_e < \sigma_c$. Thus,

$$P_s = \exp\left[-\frac{1}{4\pi} \iiint d V \, d\Omega d\sigma_c H(\sigma_e, \sigma_c) \frac{dN}{d\sigma_c} \right] \quad , \tag{6}$$

where $H(\sigma_e, \sigma_c) = 1$ when $\sigma_e > \sigma_c$

$\qquad\qquad\qquad = 0$ when $\sigma_e < \sigma_c$.

We now carry out the integral over σ_c with the result

$$P_s = \exp\left[-\frac{1}{4\pi} \iint d V \, d\Omega \, N(\sigma_e) \right] \quad . \tag{7}$$

If we assume the effective stress σ_e is simply the component of applied stress normal to a crack plane σ_n, equation (7) is essentially the same as the basic equation of Weibull's Theory B, and equation (3) becomes equivalent to the basic equation in the paper by Batdorf and Crose.[8] This means that Weibull Theory B is valid under the assumptions of Batdorf and Crose, i.e. under the approximations that cracks are non-interacting, that they are stable until such a stress is reached that they fail catastrophically, and that they are shear-insensitive, i.e. the effective stress is the normal stress σ_n.

Other assumptions concerning the fracture criterion can, of course, be made. A number of authors[9-13] have assumed that fracture occurs when the tensile stress at some point on the surface of the crack cavity exceeds the intrinsic strength of the material. The effective stress in this case depends on the crack shape and may also depend on Poisson's ratio. This approach permits evaluation of the contribution of compressive stresses to fracture. Other authors have assumed that fracture occurs when the strain energy release rate reaches a critical value.[14-17] The corresponding effective stress is readily determined for in-plane crack extension, but differences of opinion exist on how to treat the more realistic case of out-of-plane crack extension. The fracture statistics in polyaxial tension have been worked out for a number of different fracture criteria for shear-sensitive cracks by Batdorf and Heinisch.[18]

Equations (3) and (7) are alternative and completely equivalent formulations of the same theory. The physical justification of equation (3) is more readily apparent than that of equation (7), but equation (7) is preferable from a computational point of view.

FLAW DENSITY IN THEORY B

We have already discussed flaw density $n(\sigma)$ in Theory A. Weibull did not explicitly introduce flaw density into Theory B, but he did the next thing to it. In Theory A he assumed

$$P_f = 1 - \exp\left[-\int n(\sigma)\ dV\right] \tag{8}$$

with

$$n(\sigma) = k\sigma^m \tag{9}$$

In Theory B, he assumed

$$P_f = 1 - \exp\left[-\iint n_1(\sigma_n)\ dVd\Omega\right] \tag{10}$$

where

$$n_1(\sigma) = k_1\sigma^m \tag{11}$$

and the integral extends over a half-sphere. Weibull then showed that for simple tension

$$k_1 = (2m + 1)k/(2\pi) \tag{12}$$

or

$$n_1(\sigma) = (2m + 1)n(\sigma)/(2\pi) \tag{13}$$

We next seek a quantitative relation between $N(\sigma)$ and $n(\sigma)$ which we can obtain from equation (4) when we know Ω. Now Ω depends on the fracture criterion assumed. If we assume shear-insensitive cracks, then $\sigma_e = \sigma_n$, and for this case it has been shown[7] that in uniaxial tension

$$\frac{\Omega}{4\pi} = 1 - \sqrt{\frac{\sigma_c}{\sigma}} \qquad . \tag{14}$$

Inserting equation (14) into equation (4) and integrating, we obtain

$$N(\sigma) = (2m + 1)\ N_i(\sigma) = (2m + 1)n(\sigma) \qquad . \tag{15}$$

Comparing equations (13) and (15), we see that Weibull's $n_1(\sigma)$ differs only by the constant factor 2π from the $N(\sigma)$ in the present paper. He did not give it a physical interpretation, but $n_1(\sigma)$ can be regarded as the number of cracks having a critical stress less than σ that are contained in $1/(2\pi)$ units of volume. This furnishes additional evidence of the formal equivalence between Weibull's Theory B and the theory of the Batdorf and Crose paper.[8]

MATHEMATICAL EXPRESSIONS FOR CRACK DENSITY FUNCTION

More often than not, Weibull assumed that $n(\sigma)$ takes the form of a simple power law as in equation (9). This leads, in the case of a uniformly loaded specimen, to the result

$$P_f = 1 - \exp\left[- V\, n(\sigma)\right] = 1 - \exp\left[- V\, k\sigma^m\right] \quad , \tag{16}$$

In practice, a number of specimens may be tested to obtain $P_f(\sigma)$, and k and m are determined by a least squares fit to the data. The probability of failure under non-uniform or polyaxial loading is then found by inserting $n(\sigma) = k\,\sigma^m$ into Eq.(8) or Eq. (10). The result of integration is an equation of the same form as Eq. (16), in which m is unchanged but k is modified by some numberical factor. Batdorf and Heinisch[18] showed that this result holds not only for the shear-insensitive flaws considered by Weibull, but also for shear-sensitive cracks.

The use of a power law for $n(\sigma)$ implies that the weakest possible flaw has a strength of zero. Weibull also considered the possibility that the strength of the weakest flaw might not be zero but σ_u. In this case he suggested that failure probability in simple tension should be expressed by a 3-parameter relationship in the form

$$P_f = 1 - \exp\left[- V\left(\frac{\sigma - \sigma_u}{\sigma_0}\right)^m\right] = 1 - \exp\left[- V\, k(\sigma - \sigma_u)^m\right] \quad . \tag{17}$$

This introduces no serious complications for Theory A, and the probabilities of failure under variable uniaxial stresses can be given in equations of analogous form. However, there are difficulties in the case of Theory B, i.e. biaxial and triaxial stress states. Basically the problem arises from the fact that, as Vardar and Finnie have pointed out[19], the integral of $(\sigma_n - \sigma_u)^m$ over Ω called for in Eq.(7) does not result in a constant times $(\sigma - \sigma_u)^m$. Using Theory B, Dukes[19] carried out extensive calculations for various values of stress ratios and values of σ_u/σ, and expressed the results in graphical form. Evans[20] has carried out analogous calculations for pennyshaped cracks for equibiaxial and equitriaxial stress states, but using maximum strain energy release rate as the fracture criterion.

If the cumulative probability of failure in equation (15) is differentiated to get the probability density, the result within the range $0.01 \lesssim P_f \lesssim 0.99$ is a somewhat skewed Gauss curve, with a skewness determined by the value of m. Consequently, with the proper choice of m, the equation can be used to describe not only fracture statistics but also many other phenomena quite accurately, as Weibull pointed out in 1951.[21] It is not clear that Weibull attributed any fundamental physical significance to the form of equation (17), but there are some who have. Epstein[22] was among the first to point out that this equation coincides with one of the three asymptotic forms of extreme value statistics. It has been suggested[23] that all statistical fracture theories must reduce to one of these forms in the limit of large specimens (i.e. $V \to \infty$). We shall show in the next section that this is not the case. First, however, we enquire why it has appeared plausible.

Let us assume that $n(\sigma)$ can be expanded about σ_u in the rather general form

$$n(\sigma) = (\sigma - \sigma_u)^m \sum_{i=0}^{\infty} a_i (\sigma - \sigma_u)^i \tag{18}$$

where $a = k > 0$, and m is a positive number, not necessarily an integer. We now ask how $P_f(\sigma)$ behaves in a region of interest such as $0.01 < P_f < 0.99$ when V becomes very large. From Eq. (16) it follows that $0.01 < Vn(\sigma) < 4.61$. Clearly as $V \to \infty$, $n(\sigma) \to 0$, so that $(\sigma - \sigma_u) \to 0$ and $n(\sigma) \to k(\sigma - \sigma_u)^m$. Thus from Eq. (16a)

$$P_f \to 1 - \exp\left[- V k(\sigma - \sigma_u)^m \right] \tag{19}$$

Now equation (19) is the third asymptotic form of extreme value statistics. It should be noted, however, that this equation was obtained as a result of the assumption that $N(\sigma_c)$ could be expanded in a generalized Taylor series, as in equation (18). While equation (18) is plausible, it is not necessarily a valid description of crack density in real materials. In fact, as we shall show next, it probably is not.

FRACTURE STATISTICS FOR INTERGRANULAR CRACKS

McClintock[24] was the first to show that use of a physically plausible model for crack size can lead to a probability of failure that does not reduce to any of the asymptotic forms of extreme value theory. This was accomplished with the aid of a two-dimensional material model. McClintock assumed that the probability that any two adjacent grains are not bonded is some small number p, and that a long crack is simply a random aggregation of unbonded pairs of grains. He then showed that the probability that a crack is longer than x is equal to $\exp\left[-x/\lambda\right]$. From the statistics of crack size and equations of fracture mechanics, he derived the statistics of critical stress, and from this the statistics of failure. Earlier, Fisher and Holloman[25] had worked out a statistical theory of fracture for materials with randomly oriented pennyshaped cracks, using the judgmentally based assumption that the probability that a crack has a radius greater than r is equal to $\exp\left[-r/\lambda\right]$. The resulting theories were basically very similar, but the similarity is obscured by the fact that Fisher and Holloman retained the binomial product form in their calculations rather than converting to exponentials [see equation (1)]. Thus, the relation of their work to Weibull's, and the implications of their work with respect to asymptotic forms, escaped general attention. Batdorf[26] applied the McClintock hypothesis concerning crack origin to obtain crack size statistics for penny-shaped cracks, and showed that it leads to a probability of having a crack of radius greater than r that is equal to $\exp\left[-(r/a)^2\right]$. The resulting statistics for $N(\sigma_c)$ take the form

$$N(\sigma_c) = N_0 \exp\left[-A/\sigma_c^4\right] \quad , \quad (20)$$

where N_0 is the total number of cracks per unit volume and A is a somewhat complicated function of p, grain diameter, and fracture toughness. We note that $N(\sigma_c) = 0$ only when $\sigma_c = 0$, but that $N(\sigma_c)$ cannot be expanded as a Maclaurin series about $\sigma_c = 0$ because at that point all its derivatives vanish. Thus, we would not expect $P_f(\sigma)$, as given by this crack density function, to be asymptotically stable and, in fact, it is not. The theory leads to the following result for

small failure probabilities

$$P_f = 1 - \exp\left[- B_1 \sigma^4 e^{- B_2/\sigma^4}\right] \quad . \tag{20}$$

The deviations between this theory and Weibull theory are not as large as one might expect from the dramatically different equations for the density function and the probability of failure. Probability of failure curves have been constructed for various values of the parameters (N_o, p, K_{Ic}, etc.), and it has been shown that use of Weibull's 3-parameter equation leads to a good fit over a limited range of failure probability (such as $0.01 \leq P_f \leq 0.99$). However, the Weibull parameters σ_u and m giving the best fit are weak functions of specimen volume instead of being volume-independent as in Weibull theory. In addition, the two classes of theory differ markedly in their extrapolation to very low values of $P_f(\sigma)$.

It is not suggested, of course, that the theory for intergranular cracks just described is an accurate portrayal of physical reality. Several approximations are made, such as the assumptions that neighboring grains are either perfectly bonded or completed unbonded, that cracks are non-interacting and pennyshaped, etc. Future work will undoubtedly lead to improvements in the model. The point to be emphasized, however, is that an improved theory can be expected to differ from Weibull theory in much the same manner as the theory just outlined does. It seems likely that physical reality differs from Weibull theory in a somewhat similar way, but possibly by a small enough amount to make it difficult to establish these differences experimentally with complete confidence.

CONCLUDING REMARKS

Weibull was the first to apply the weakest link concept to brittle frac-
ture. He showed that any weakest link theory must result in a failure
probability for a body uniformly stressed in simple tension in the form

$$P_f = 1 - \exp\left[- V n(\sigma)\right] \qquad , \qquad (23)$$

where $n(\sigma)$ is a positive and monotonic, but otherwise arbitrary, function
of the applied stress. He proposed two computationally convenient repre-
sentations for $n(\sigma)$, one involving two parameters and one involving three.
These are so widely used that often experimentalists present their data
simply by stating the values of the Weibull parameters leading to the best
fit.

Weibull also showed how to apply his theory to predict the probability
of fracture in arbitrarily stressed bodies when the failure statistics for
simple tension are known. His work was primarily founded on the consider-
ations of extreme value theory, with no explicit attention to either fracture
mechanics or material microstructure.

In the case of uniaxial stress states, Weibull assumed that only tension
contributes to fracture. As a result, it was not necessary to adopt a
fracture criterion. This theory is basically valid for tensile type fractures,
subject however to certain limitations. Use of either
the 2- or the 3-parameter Weibull representation of fracture statistics in a
stress range for which test data are not available is unsafe. This means we
have no justification for extrapolation to very low probability of failure or to
specimens very much larger or smaller than those tested. The hazard
involved in such extrapolations is made especially clear by the results of
the theory for the fracture statistics of intergranular cracks discussed in the
previous section.

To deduce the polyaxial stress fracture statistics from uniaxial data,
it is necessary to specify a fracture criterion. Weibull did not do this
explicitly, but his recommended procedure implies the assumption that

fracture occurs when the component of tensile stress normal to a crack plane exceeds some critical value.

Recent improvements in weakest link theory include incorporation of more accurate fracture criteria and explorations of the implications of fracture mechanics and material microstructure. While these developments are important theoretical advances, it is too early to assess fully how much they will improve fracture statistics from a practical point of view. In any event, it remains an impressive tribute to Weibull's technical grasp and physical insight that the world has required several decades to discover how to advance fracture statistics significantly beyond the point he had already reached in 1939.

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